Chaotic Vector Fields and Commutators

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We discuss the question whether the commutator of two chaotic vector fields is again a chaotic vector field.

Key words: Chaos; Vector Fields; Commutators; Divergence.

We consider autonomous systems of first-order ordinary differential equations

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = f_j(\mathbf{u}), \qquad j = 1,2,3$$

with chaotic behaviour [1]. We assume that the $f_j : \mathbf{R}^3 \to \mathbf{R}$ are analytic functions. Associated with this autonomous first-order system is the analytic vector field

$$V = f_1(\mathbf{u})\frac{\partial}{\partial u_1} + f_2(\mathbf{u})\frac{\partial}{\partial u_2} + f_3(\mathbf{u})\frac{\partial}{\partial u_3}.$$

The analytic vector fields form a Lie algebra under the commutator [2]. Typical examples are the Lorenz model, the Rikitake two-disc dynamo system, and Chen's model [3]. Other simple examples together with their three one-dimensional Ljapunov exponents are listed by Sprott [4, 5]. For the three examples given above the divergence of the corresponding vector field is a negative constant. Thus a volume element will shrink under the flow.

Consider two analytic vector fields, say V and W, with chaotic behaviour. Then an interesting question is whether the vector field we obtain from the commutator is again chaotic. With

$$W = g_1(\mathbf{u})\frac{\partial}{\partial u_1} + g_2(\mathbf{u})\frac{\partial}{\partial u_2} + g_3(\mathbf{u})\frac{\partial}{\partial u_3}$$

we obtain the vector field

$$[V,W] = \sum_{i=1}^{3}\sum_{j=1}^{3}\left(f_i\frac{\partial g_j}{\partial u_i}\frac{\partial}{\partial u_j} - g_j\frac{\partial f_i}{\partial u_j}\frac{\partial}{\partial u_i}\right).$$

Let $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ be the volume form in $\mathbf{R}^3$. Then $L_V\omega = (\mathrm{div}V)\omega$, where $L_V(.)$ denotes the Lie derivative and $\mathrm{div}V$ denotes the divergence of the vector field V. We find the divergence of the vector field given by the commutator $[V,W]$. Using the property of the Lie derivative such as linearity and the product rule we find

$$L_{[V,W]}\omega = L_V(L_W\omega) - L_W(L_V\omega)$$
$$= (L_V(\mathrm{div}W) - L_W(\mathrm{div}V))\omega.$$

Since $\mathrm{div}V$ for the first system is constant and $\mathrm{div}W$ for the second system is also constant, we find $L_{[V,W]}\omega = 0$. Thus the vector field given by the commutator is divergenceless.

Thus the divergence of the vector field given by the commutator is zero if the divergence of the two vector fields V and W is a (negative) constant. Typical examples for vector fields with a negative constant divergence are the Lorenz model, Rikitake two-disc dynamo system and Chen's model. If λ_1^I, λ_2^I, λ_3^I are the three one-dimensional Ljapunov exponents with the ordering $\lambda_1^I \ge \lambda_2^I \ge \lambda_3^I$, then the classification would be $\lambda_1^I < 0$, $\lambda_2^I < 0$, $\lambda_3^I < 0$ (fixed point), $\lambda_1^I = 0$, $\lambda_2^I < 0$, $\lambda_3 < 0$ (limit cycle), $\lambda_1^I = 0$, $\lambda_2^I = 0$, $\lambda_3 < 0$ (torus), $\lambda_1^I > 0$, $\lambda_2^I = 0$, $\lambda_3^I < 0$ (chaos). If $\mathrm{div}V = c$, where c is a negative constant, we have

$$\lambda_1^I + \lambda_2^I + \lambda_3^I = c.$$

Now we proved that, if $\mathrm{div}V = c_1$ and $\mathrm{div}W = c_2$ are (negative) constants, we have for the vector field $[V,W]$

$$\lambda_1^I + \lambda_2^I + \lambda_3^I = 0.$$

Since for a chaotic system $\lambda_2^I = 0$ we obtain $\lambda_1^I = -\lambda_3^I$. A simple example of a dynamical system with chaotic behaviour and divergence equal to 0 is

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_2, \quad \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_3, \quad \frac{\mathrm{d}u_3}{\mathrm{d}t} = -u_2 + u_1^2 - \frac{1}{100}.$$

Here we also find unbounded trajectories.

Obviously we cannot expect that the commutator of any two chaotic vector fields provides a chaotic vector field. For example consider the autonomous system

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_2u_3, \quad \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_1 - u_2, \quad \frac{\mathrm{d}u_3}{\mathrm{d}t} = a - u_1u_2.$$

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For $a = 1$ (and values close to $a = 1$) we have chaotic behaviour with the one-dimensional Ljapunov exponents $\lambda_1^I = 0.21$, $\lambda_2^I = 0$, $\lambda_3^I = -1.21$. Take the same vector field except the constant a replaced by b. Then the commutator provides the linear vector field

$$(a-b)u_2\frac{\partial}{\partial u_1}.$$

One could argue that the two vector fields are "to close". An example where the commutator provides a new chaotic system is the commutator of the Lorenz model:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \sigma(u_2 - u_1), \quad \frac{\mathrm{d}u_2}{\mathrm{d}t} = ru_1 - u_2 - u_1u_3,$$
$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = -bu_3 + u_1u_2,$$

and of Chen's model:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = a(u_2 - u_1), \quad \frac{\mathrm{d}u_2}{\mathrm{d}t} = du_1 - u_1u_3 + cu_2,$$
$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = u_1u_2 - bu_3.$$

The Lorenz model shows chaotic behaviour for the parameters $\sigma = 10$, $r = 28$, $b = 8/3$. Chen's model shows chaotic behaviour for $a = 36$, $b = 3$, $c = 28$. Studying the vector field of the commutator of these vector fields provides a vector field with divergence 0 and chaotic behaviour. However, we also find unbounded trajectories.

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